

University of Groningen

Self-dual supergravity theories in 2 + 2 dimensions

Bergshoeff, E; Sezgin, E

Published in:
Physics Letters B

DOI:
[10.1016/0370-2693\(92\)90612-8](https://doi.org/10.1016/0370-2693(92)90612-8)

IMPORTANT NOTE: You are advised to consult the publisher's version (publisher's PDF) if you wish to cite from it. Please check the document version below.

Document Version
Publisher's PDF, also known as Version of record

Publication date:
1992

[Link to publication in University of Groningen/UMCG research database](#)

Citation for published version (APA):

Bergshoeff, E., & Sezgin, E. (1992). Self-dual supergravity theories in 2 + 2 dimensions. *Physics Letters B*, 292(1-2), 87-92. [https://doi.org/10.1016/0370-2693\(92\)90612-8](https://doi.org/10.1016/0370-2693(92)90612-8)

Copyright

Other than for strictly personal use, it is not permitted to download or to forward/distribute the text or part of it without the consent of the author(s) and/or copyright holder(s), unless the work is under an open content license (like Creative Commons).

The publication may also be distributed here under the terms of Article 25fa of the Dutch Copyright Act, indicated by the "Taverne" license. More information can be found on the University of Groningen website: <https://www.rug.nl/library/open-access/self-archiving-pure/taverne-amendment>.

Take-down policy

If you believe that this document breaches copyright please contact us providing details, and we will remove access to the work immediately and investigate your claim.

Downloaded from the University of Groningen/UMCG research database (Pure): <http://www.rug.nl/research/portal>. For technical reasons the number of authors shown on this cover page is limited to 10 maximum.

Self-dual supergravity theories in 2+2 dimensions

Eric Bergshoeff

Institute for Theoretical Physics, University of Groningen, Nijenborgh 4, NL-9747 AG Groningen, The Netherlands

and

Ergin Sezgin¹

Center for Theoretical Physics, Texas A&M University, College Station, TX 77843-4242, USA

Received 20 July 1992

Starting from the new minimal multiplet of supergravity in 2+2 dimensions, we construct two types of self-dual supergravity theories. One of them involves a self-duality condition on the Riemann curvature and implies the equations of motion following from the Hilbert–Einstein type supergravity action. The other one involves a self-duality condition on a *torsionful* Riemann curvature with the torsion given by the field-strength of an antisymmetric tensor field, and implies the equations of motion that follow from an R^2 -type action.

1. Introduction

Self-dual supergravity theories in 2+2 dimensions are interesting to study for at least two reasons. First, they may arise as consistent backgrounds for the $N=2$ superstring theories [1]. Second, their suitable reductions to 1+1 dimensions are expected to yield a large class of integrable systems^{#1}. Recently, a self-dual supergravity in 2+2 dimensions has been constructed [3]. It is essentially obtained by imposing self-duality and chirality conditions on the curvatures of an Hilbert–Einstein type supergravity theory.

In this letter we shall construct a new type of self-dual supergravity in 2+2 dimensions. Our construction is based upon the new minimal multiplet of supergravity [4] in which the antisymmetric tensor component of the off-shell supergravity multiplet occurs as the torsion part of the spin connection field [5]. In this theory one arrives naturally at a self-duality condition on the *torsionful* Riemann tensor. We will show that in this type of self-dual supergrav-

ity theory the field equations of an R^2 -type action must be satisfied.

It has been shown in refs. [5,6] that supergravity in the new minimal formulation can be put into a form which is very similar to the super Yang–Mills system with the Yang–Mills field replaced by a Lorentz connection with torsion. We shall therefore begin with a discussion of self-dual Yang–Mills theory [3]. We shall show that this theory actually has a hidden superconformal symmetry which is based on the superalgebra $SL(4|1)$. We shall then describe our construction of the two types of self-dual supergravity theories in 2+2 dimensions. In the conclusion we will discuss a number of questions raised by this work. In the appendix we give details of the superconformal algebra $SL(4|1)$ in 2+2 dimensions.

2. Self-dual superconformal Yang–Mills theory

We start with the description of the super-Poincaré algebra in 2+2 dimensions. The generators are the supercharges Q_{\pm} , which are pseudo-Majorana–Weyl spinors in 2+2 dimensions [7], and the Poincaré

¹ Supported in part by the US National Science Foundation, under grant PHY-9106593.

^{#1} See e.g. the review article [2].

generators, P_μ and $M_{\mu\nu}$. The anticommutators between the supercharges are given by ^{#2}

$$\begin{aligned}\{Q_+, Q_+\} &= 0, \\ \{Q_+, Q_-\} &= \frac{1}{2}(1 + \gamma_5)\gamma_\mu CP^\mu, \\ \{Q_-, Q_-\} &= 0.\end{aligned}\quad (1)$$

From the well-known super Yang–Mills theory in 3+1, it is easy to extract the analogous result for 2+2 dimensions, which reads

$$\delta A_\mu = \frac{1}{2}\bar{\epsilon}_+ \gamma_\mu \lambda_- + \frac{1}{2}\bar{\epsilon}_- \gamma_\mu \lambda_+, \quad (2)$$

$$\delta \lambda_\pm = -\frac{1}{4}\gamma^{\mu\nu} F_{\mu\nu} \epsilon_\pm \pm \frac{1}{2} D \epsilon_\pm, \quad (3)$$

$$\delta D = \frac{1}{2}\bar{\epsilon}_+ \gamma^\mu D_\mu \lambda_- - \frac{1}{2}\bar{\epsilon}_- \gamma^\mu D_\mu \lambda_+, \quad (4)$$

where λ_\pm are pseudo-Majorana–Weyl spinors, D is the real auxiliary scalar field and D_μ is the gauge-covariant derivative. All fields carry an adjoint Yang–Mills index which we have suppressed. Since this is an off-shell multiplet, no field equations, including the self-duality condition, are implied by the above transformation rules. Following ref. [3] we now impose the condition

$$\lambda_+ = 0. \quad (5)$$

The consequences of this condition are as follows. First from (3) we obtain $D=0$ and the self-dual Yang–Mills equation

$$\gamma^{\mu\nu} \epsilon_+ F_{\mu\nu} = 0 \Leftrightarrow F_{\mu\nu} = \frac{1}{2} \epsilon_{\mu\nu\rho\sigma} F^{\rho\sigma}. \quad (6)$$

Next, from (4) we deduce that the nonvanishing spinor λ_- satisfies the field equation

$$\gamma^\mu D_\mu \lambda_- = 0. \quad (7)$$

In summary, the self-dual super Yang–Mills system simply consists of the field equations (6) and (7) which transform into each other under the supersymmetry transformations

$$\delta A_\mu = \frac{1}{2}\bar{\epsilon}_+ \gamma_\mu \lambda_-, \quad (8)$$

^{#2} Our conventions are $\{\gamma_\mu, \gamma_\nu\} = 2\eta_{\mu\nu}$, where $\eta_{\mu\nu} = \text{diag}(-, -, +, +)$ and $\gamma_5^2 = \gamma_5$; $C = \gamma_{12}$, $C^T = -C$, $(\gamma^\mu C)^T = (\gamma^\mu C)$, $(\gamma^{\mu\nu} C)^T = \gamma^{\mu\nu} C$, $\epsilon_{1234} = +1$; $\gamma_5 = \gamma_{1234}$, $\gamma_5^2 = 1$, $[C, \gamma_5] = 0$. Q_\pm are real two-component independent pseudo-Majorana–Weyl spinors. The Fierz rearrangement formula is $\chi\bar{\psi} = -\frac{1}{2}\bar{\psi}\chi + \frac{1}{4}\bar{\psi}\gamma^{\mu\nu}\chi\gamma_{\mu\nu}$ for the same chirality anticommuting pseudo-Majorana–Weyl spinors and $\chi\bar{\psi} = -\frac{1}{2}\bar{\psi}\gamma^\mu\chi\gamma_\mu$ for the case of opposite chiralities. A useful identity is $\gamma^{\mu\nu} = -\frac{1}{2}\epsilon^{\mu\nu\rho\sigma}\gamma_{\rho\sigma}\gamma_5$.

$$\delta \lambda_- = -\frac{1}{4}\gamma^{\mu\nu} F_{\mu\nu} \epsilon_-. \quad (9)$$

One can check that the closure of the algebra indeed requires (6) and (7). Note that both supersymmetries Q_\pm are present in this system and therefore we have (1, 1) type supersymmetry ^{#3}. Notice that despite supersymmetry, the self-duality equation (6) receives no fermionic modifications.

It is well-known that the self-dual Yang–Mills equations are invariant under the conformal group of transformations, which is the group $SO(3, 3)$ for a spacetime with signature (2, 2). Not surprisingly, the self-dual super Yang–Mills equations (6) and (7) are invariant under the superconformal group $SL(4|1)$ whose bosonic subgroup is $SL(4) \approx SO(3, 3)$. These transformations are given by (8), (9) and

$$\delta A_\mu = -\frac{1}{2}\bar{\eta}_- \gamma^\nu \gamma_\mu \lambda_- x_\nu + \xi^\lambda \partial_\lambda A_\mu, \quad (10)$$

$$\begin{aligned}\delta \lambda_- &= -\frac{1}{4}\gamma^{\mu\nu}\gamma^\rho x_\rho F_{\mu\nu} \eta_+ + \frac{3}{2}\alpha \lambda_- - \beta \lambda_- \\ &+ \frac{1}{4}\omega^{\mu\nu} \gamma_{\mu\nu} \lambda_- + \xi^\mu \partial_\mu \lambda_-, \end{aligned}\quad (11)$$

where η_\pm are the special conformal supersymmetry parameters, α, β are the parameters of dilatations and $SO(1, 1)$ transformations, respectively, and ξ^μ represent the spacetime conformal transformations given by

$$\xi^\mu(x) = a^\mu - \omega^{\mu\nu} x_\nu + \alpha x^\mu + 2x^\mu \eta \cdot x - \eta^\mu x^2. \quad (12)$$

Here a^μ , $\omega^{\mu\nu}$ and η^μ are the constant parameters of translations, Lorentz rotations and conformal boosts, respectively. The above superconformal transformations indeed satisfy the superconformal algebra $SL(4|1)$ given in the appendix.

3. Self-dual supergravity theory

In 3+1 dimensions there are many different versions of off-shell supergravity. Similar off-shell supergravities in 2+2 dimensions can be easily obtained from them. We shall work with the new

^{#3} The truncation to (1, 0) supersymmetry by setting $\epsilon_+ = 0$ would lead to the superalgebra $\{Q_-, Q_-\} = 0$ which is realized trivially by the transformation rules $\delta A_\mu = 0$ and $\delta \lambda_- = -\frac{1}{4}\gamma^{\mu\nu} F_{\mu\nu} \epsilon_-$. The truncation to (0, 1) supersymmetry which is achieved by setting $\epsilon_- = 0$ would lead to the superalgebra $\{Q_+, Q_+\} = 0$, again realized trivially by the transformation rules $\delta \lambda_- = 0$ and $\delta A_\mu = \frac{1}{2}\bar{\epsilon}_+ \gamma_\mu \lambda_-$.

minimal formulation. As has been shown in refs. [5,6] this supergravity theory can be put into a form which is very similar to the super Yang–Mills system with the Yang–Mills field replaced by a Lorentz connection with torsion. This is a convenient feature since it allows us to apply the procedure of the previous section to obtain self-dual supergravity equations.

The new minimal multiplet [4] contains the vierbein e_μ^a , the gravitini $\psi_{\mu\pm}$, the antisymmetric tensor field $B_{\mu\nu}$ and an $SO(1,1)$ gauge field V_μ . In 2+2 dimensions the supersymmetry transformations are

$$\delta e_\mu^a = \frac{1}{2}\bar{\epsilon}_+ \gamma^a \psi_{\mu-} + \frac{1}{2}\bar{\epsilon}_- \gamma^a \psi_{\mu+}, \quad (13)$$

$$\delta \psi_{\mu\pm} = \mathcal{D}_\mu(\Omega_\pm, V_\pm) \epsilon_\pm, \quad (14)$$

$$\delta B_{\mu\nu} = \frac{3}{2}\bar{\epsilon}_+ \gamma_{[\mu} \psi_{\nu]-} + \frac{3}{2}\bar{\epsilon}_- \gamma_{[\mu} \psi_{\nu]+}, \quad (15)$$

$$\delta V_\mu = \frac{1}{8}\bar{\epsilon}_+ \gamma_\mu \gamma^{ab} \psi_{ab-} - \frac{1}{8}\bar{\epsilon}_- \gamma_\mu \gamma^{ab} \psi_{ab+}, \quad (16)$$

where we have defined the combinations ^{#4}

$$\Omega_{\mu\pm}^{ab} = \omega_\mu^{ab} = \omega(e, \psi) \pm \hat{H}_\mu^{ab}, \quad (17)$$

$$V_{\mu+} = V_\mu + \frac{1}{6}\epsilon_\mu^{abc} \hat{H}_{abc}. \quad (18)$$

The covariant curvatures are

$$\psi_{\mu\nu\pm} = \mathcal{D}_\mu(\Omega_\pm, V_\pm) \psi_{\nu\pm} - \mathcal{D}_\nu(\Omega_\pm, V_\pm) \psi_{\mu\pm}, \quad (19)$$

$$\hat{H}_{\mu\nu\rho} = \partial_{[\mu} B_{\nu\rho]} - \frac{3}{2}\bar{\psi}_{[\mu} \gamma_{\nu} \psi_{\rho]-}. \quad (20)$$

The derivative \mathcal{D}_μ on the supersymmetry parameters ϵ_\pm is given by

$$\mathcal{D}_\mu(\Omega_\pm, V_\pm) \epsilon_\pm = (\partial_\mu - \frac{1}{4}\Omega_{\mu+}^{ab} \gamma_{ab} \mp V_{\mu+}) \epsilon_\pm. \quad (21)$$

Following refs. [5,6], one can show that $\Omega_{\mu-}^{ab}$, ψ^{ab} and the supercovariant field-strength $\hat{F}^{ab}(V_+)$ form a super Yang–Mills type multiplet with the gauge group taken to be the spacetime Lorentz group $SO(2,2)$. We find that the transformation rules are ^{#5}

^{#4} The supercovariant spin-connection $\omega_\mu^{ab}(e, \psi)$ is defined as follows:

$$\omega_\mu^{ab}(e, \psi) = \omega_\mu^{ab}(e) - \frac{1}{2}\bar{\psi}_\mu \gamma^{[a} \psi^{b]} - \frac{1}{4}\bar{\psi}^a \gamma_\mu \psi^b,$$

where $\psi_\mu = \psi_{\mu+} + \psi_{\mu-}$.

^{#5} The supercovariant curvatures of $V_{\mu+}$ and $\Omega_{\mu-}^{ab}$ are defined as follows:

$$\hat{F}_{ab}(V_+) = 2\partial_{[a} V_{b]+} - \frac{1}{2}\bar{\psi}_{[a} \gamma^{\lambda} \psi_{b]\lambda-} + \frac{1}{2}\bar{\psi}_{[a-} \gamma^{\lambda} \psi_{b]\lambda+},$$

$$\hat{R}_{cd}^{ab}(\Omega_-) = R_{cd}^{ab}(\Omega_-) - \bar{\psi}_{[c+} \gamma_d] \psi_-^{ab} - \bar{\psi}_{[c-} \gamma_d] \psi_+^{ab}.$$

$$\delta \Omega_{\mu-}^{ab} = \frac{1}{2}\bar{\epsilon}_+ \gamma_\mu \psi_-^{ab} + \frac{1}{2}\bar{\epsilon}_- \gamma_\mu \psi_+^{ab}, \quad (22)$$

$$\delta \psi_\pm^{ab} = -\frac{1}{4}\gamma^{cd} \hat{R}_{cd}^{ab}(\Omega_\pm) \epsilon_\pm \mp \hat{F}^{ab}(V_\pm) \epsilon_\pm, \quad (23)$$

$$\delta \hat{F}^{ab}(V_+) = -\frac{1}{4}\bar{\epsilon}_+ \gamma^\mu \mathcal{D}_\mu(\Omega_-) \psi_-^{ab} + \frac{1}{4}\bar{\epsilon}_- \gamma^\mu \mathcal{D}_\mu(\Omega_-) \psi_+^{ab}, \quad (24)$$

where $\mathcal{D}_\mu(\Omega_-)$ is the supercovariant derivative and the Ω_- covariantization in the last line acts on the fermionic as well as the vectorial indices of ψ_\pm^{ab} .

Below we shall construct two types of self-dual supergravity theories which we shall refer to as type I and type II. To obtain the type I self-dual supergravity we shall exploit the fact that the transformation rules (22)–(24) have the super Yang–Mills form and use the procedure of the previous section. This leads to the self-dual supergravity theory of ref. [3]. To obtain the type II self-dual supergravity we shall make use of the fact that the Yang–Mills group in (22)–(24) is actually the spacetime group $SO(2,2)$.

3.1. Type I reduction

Following the procedure of the previous section we impose the condition

$$\psi_+^{ab} = 0. \quad (25)$$

The consequence of this can be seen from (23) and (24) to be

$$\hat{R}_{cd}^{ab}(\Omega_-) = \frac{1}{2}\epsilon_{cdef} \hat{R}^{efab}(\Omega_-), \quad (26)$$

$$\hat{F}^{ab}(V_+) = 0, \quad (27)$$

$$\gamma^\mu \mathcal{D}_\mu(\Omega_-) \psi_-^{ab} = 0. \quad (28)$$

We recognize (27) as one of the equations of motion that follow from the 2+2 version of the R -type action of ref. [4],

$$e^{-1} \mathcal{L}(R) = -\frac{1}{2}R(\omega) - \bar{\psi}_{\mu+} \gamma^{\mu\nu\rho} \mathcal{D}_\nu(\omega, V) \psi_{\rho-} - \frac{1}{2}\hat{H}_{abc} \hat{H}^{abc} + \frac{2}{3}e^{-1}\epsilon^{\mu\nu\lambda\rho} V_\mu \partial_\nu B_{\lambda\rho}. \quad (29)$$

By supersymmetry, the remaining equations of motion based on this action also follow. As for (28), it can be written in terms of derivatives of the gravitino and Einstein equations of motion.

One of the equations of motion that follows from (29) is

$$\hat{H}_{abc} = 0. \quad (30)$$

Hence, (26) reduces to

$$R_{cd}{}^{ab}(\omega(e)) = \frac{1}{2}\epsilon_{cdef}R^{efab}(\omega(e)), \quad (31)$$

where we have used (25) ^{#6}. From (31) and the identity $R_{[abc]d}(\omega) = 0$, it then follows that the Ricci-tensor vanishes, i.e.

$$R_{ab}(\omega) = 0. \quad (32)$$

The supersymmetry variation of (31) leads to the following duality condition on the gravitino curvature:

$$\psi_-^{ab} = \frac{1}{2}\epsilon^{abcd}\psi_{cd-}. \quad (33)$$

This duality condition can also directly be proven from the gravitino field equation $\gamma_a\psi_-^{ab} = 0$. The same gravitino field equation, together with the condition (25) is equivalent to the gravitino curvature constraint imposed in ref. [3]. Thus, our component results for the type I self-dual supergravity are presumably equivalent to the superspace results of ref. [3].

In summary, the type I self-dual supergravity is described by (31) and (33) plus the Bianchi identities for the curvatures which together imply the equations of motion that follow from the R action. Due to (25), (27) and (30) the fields $\psi_{\mu+}$, V_{μ} and $B_{\mu\nu}$ drop out of the theory and the only equations of motions in addition to (31) and (33) are the supercovariant Einstein and gravitino field equations for the fields $e_{\mu}{}^a$ and $\psi_{\mu-}$, respectively. Thus, the type I self-dual supergravity has the local supersymmetry given by

$$\delta e_{\mu}{}^a = \frac{1}{2}\epsilon_+ \gamma^a \psi_{\mu-}, \quad (34)$$

$$\delta \psi_{\mu-} = \mathcal{D}_{\mu}(\omega)\epsilon_-. \quad (35)$$

Closure of the algebra requires that the supersymmetry parameter ϵ_+ be covariantly constant, i.e.

$$D_{\mu}(\omega)\epsilon_+ = 0. \quad (36)$$

The integrability condition of this equation leads to the self-duality condition (31) for the Riemann curvature.

Note that despite that fact that there are no fermionic corrections to the self-duality equation (31)

^{#6} We have solved (25) by choosing the gauge $\psi_{\mu+} = 0$. In this gauge, all fermionic terms in the spin connection $\omega(e, \psi)$ and the supercovariant curvature $\hat{R}_{cd}{}^{ab}$ vanish. As a consequence, the self-duality condition (31) does not contain any fermionic terms.

it nonetheless is supercovariant and transforms into the gravitino field equation under the above supersymmetry transformations. Note also that, since the self-duality conditions (31) and (33) do not follow from the action (29), of course the latter cannot be considered as the action for type I self-dual supergravity.

3.2. Type II reduction

In this reduction we make use of the fact that each one of the gravitino curvatures ψ_{\pm}^{ab} is in a reducible representation of $SO(2, 2)$, namely $(3, 1) \oplus (1, 3)$. The type II reduction amounts to setting one of these pieces equal to zero, i.e.

$$\psi_{\pm}^{ab} = \frac{1}{2}\epsilon^{abcd}\psi_{cd\pm}. \quad (37)$$

This condition and supersymmetry are evidently consistent with the following self-duality conditions:

$$\hat{R}_{cd}{}^{ab}(\Omega_-) = \frac{1}{2}\epsilon^{abef}\hat{R}_{cdef}(\Omega_-), \quad (38)$$

$$\hat{F}^{ab}(V_+) = \frac{1}{2}\epsilon^{abcd}\hat{F}_{cd}(V_+). \quad (39)$$

The above three equations, together with the Bianchi identities for the curvatures, define type II self-dual supergravity. We observe that (39) together with the Bianchi identity

$$D_{[a}\hat{F}_{bc]}(V_+) + \frac{1}{4}\bar{\psi}_{[ab+}\gamma^{\lambda}\psi_{c]\lambda-} - \frac{1}{4}\bar{\psi}_{[ab-}\gamma^{\lambda}\psi_{c]\lambda+} = 0 \quad (40)$$

and the self-duality conditions (37) imply that

$$8D_a\hat{F}^{ab}(V_+) + \bar{\psi}_+^{cd}\gamma^b\psi_{cd-} = 0. \quad (41)$$

We recognize this as one of the field equations that follow from the 2+2 version of the R^2 -type supergravity action constructed in refs. [5,6],

$$\begin{aligned} e^{-1}\mathcal{L}(R^2) = & \frac{1}{4}R_{\mu\nu}{}^{ab}(\Omega_-)R^{\mu\nu ab}(\Omega_-) \\ & - 2\hat{F}_{ab}(V_+)\hat{F}^{ab}(V_+) + \frac{1}{2}\bar{\psi}^{ab}\gamma^{\mu}D_{\mu}(\Omega_-, V)\psi_{ab} \\ & + \frac{1}{8}\bar{\psi}_{\mu}\gamma^{cd}\gamma^{\mu}\psi_{ab}[\hat{R}_{cd}{}^{ab}(\Omega_-) + \hat{R}_{cd}{}^{ab}(\Omega_-)], \end{aligned} \quad (42)$$

where $\psi_{\mu} = \psi_{\mu+} + \psi_{\mu-}$ and the Ω_- covariantization in the kinetic term of the gravitino curvature acts both on the spinor as well as the vector indices of ψ^{ab} . By supersymmetry it then follows that the field equations for the remaining fields $e_{\mu}{}^a$, $\psi_{\mu\pm}$ and $B_{\mu\nu}$ must also be satisfied. We arrive at the same conclusion by

considering the consequences of (37) and (38) together with the Bianchi identities.

In summary, the type II self-dual supergravity theory consists of the self-duality equations (37)–(39) plus the Bianchi identities for the curvatures which together imply the equations of motion that follow from the R^2 action. Note that in contradistinction to the type I theory the self-duality condition on the Riemann curvature involves extra fermionic terms. The type II theory is invariant under the following supersymmetry transformations:

$$\delta e_\mu^a = \frac{1}{2} \bar{\epsilon}_+ \gamma^a \psi_{\mu-} + \frac{1}{2} \bar{\epsilon}_- \gamma^a \psi_{\mu+}, \quad (43)$$

$$\delta \psi_{\mu\pm} = (\partial_\mu - \frac{1}{4} \Omega_{\mu+}{}^{ab} \gamma_{ab} \mp V_{\mu+}) \epsilon_\pm, \quad (44)$$

$$\delta B_{\mu\nu} = \frac{3}{2} \bar{\epsilon}_+ \gamma_{[\mu} \psi_{\nu]-} + \frac{3}{2} \bar{\epsilon}_- \gamma_{[\mu} \psi_{\nu]+}, \quad (45)$$

$$\delta V_\mu = \frac{1}{8} \bar{\epsilon}_+ \gamma_\mu \gamma^{ab} \psi_{ab-}. \quad (46)$$

Although the equations of motion of type II self-dual supergravity imply the equations of motion corresponding to the above R^2 action, the converse is not true and therefore the R^2 action, of course, cannot be considered as the action for the type II self-dual supergravity.

Finally, we note that the self-duality condition (38) implies that

$$\hat{R}_{cd}{}^{ab}(\Omega_+) = \frac{1}{2} \epsilon_{cdef} \hat{R}{}^{efab}(\Omega_+) \quad (47)$$

owing to the identity

$$\hat{R}_{abcd}(\Omega_-) = \hat{R}_{cdab}(\Omega_+). \quad (48)$$

Furthermore, for a *torsionful* curvature the self-duality condition (38) does *not* imply that the Ricci tensor vanishes. Instead, one is only able to show that the Ricci tensor is proportional to torsion-dependent terms.

4. Comments

In this letter we have constructed a new type of self-dual supergravity theory in 2+2 dimensions based on a *torsionful* curvature tensor. Our results raise a number of interesting questions.

A natural question to ask is the connection between the self-dual supergravities considered here and the $N=2$ supersymmetric string theories. The appropriate ones to consider are the “type II” $N=2$ super-

string theories. They have different versions labeled by a positive integer and are called Z_n -symmetric strings [1]. The simplest version ($n=0$) has ordinary self-dual gravity as a consistent background, and it describes a spin-0 particle. In general, the Z_n -string has n particles with spins $0, 1/n, 2/n, \dots, (n-1)/n$. Thus, the Z_2 string seems to be the best candidate to have a connection with the self-dual supergravity theories described above^{#7}.

In search of spacetime supersymmetric versions of $N=2$ superstrings it may be useful to consider their Green–Schwarz type formulations. In this regard we note that *off-shell* supergravity in the new minimal formulation is a consistent background for the Green–Schwarz superstring in 3+1 dimensions [8]. The same model can be taken over to represent the Green–Schwarz superstring in 2+2 dimensions [3], with obvious changes in the signature of the spacetime metric and the reality conditions on the spinorial coordinates of the target superspace, which will now be parametrized by x^μ , θ_+ and θ_- . It would be interesting to find a mechanism that will put the supergravity background on-shell, with or without self-duality conditions, and to find the “ $N=2$ supersymmetric” versions of these Green–Schwarz type superstring theories.

It is also worth mentioning that the mathematics used in discussing self-dual Yang–Mills or supergravity theories resembles that used in the discussion of Kaluza–Klein compactifications and solitonic solutions to string theories in ten dimensions. An important difference, however, is that in the latter case all background fermions are set equal to zero and not all supersymmetries are maintained, depending on the background. Typically, residual supersymmetries are found in self-dual backgrounds. In the case of self-dual supergravities all supersymmetries are maintained and not all fermions are vanishing.

Finally, we recall that the supergravities considered in this letter have a $(1, 1)$ supersymmetry in 2+2 dimensions. A chiral truncation to $(1, 0)$ supersymmetry would lead to the superalgebra $\{Q_-, Q_-\} = 0$. In this case there is no need to supersymmetrize the bosonic self-duality conditions since the gauge fields will be inert under supersymmetry. To obtain a non-trivial $(1, 0)$ -supersymmetric self-dual system in 2+2

^{#7} We thank C. Vafa for pointing this to us.

dimensions, one should consider the following superalgebra:

$$\{Q_-, Q_-\} = \frac{1}{2}\gamma^{\mu\nu}CM_{\mu\nu}, \quad (49)$$

where $M_{\mu\nu}$ are the Lorentz generators, self-dual in the μ, ν indices, therefore representing effectively an $SO(2, 1)$ rotation. A $(1, 0)$ self-dual supergravity theory based upon this algebra has been considered in ref. [9].

Acknowledgement

We are grateful to Larry Romans, Mees de Roo and Cumrun Vafa for stimulating discussions and we thank Chris Pope for a careful reading of the manuscript. E.S. would like to thank the Institute for Theoretical Physics in Groningen and the International Center for Theoretical Physics in Trieste for hospitality. For one of us (E.B.) this work has been made possible by a fellowship of the Royal Netherlands Academy of Arts and Sciences (KNAW).

Appendix.

The superconformal algebra $SL(4|1)$

The bosonic subalgebra of $SL(4|1)$ is $SL(4) \times SO(1, 1)$ generated by the $SO(2, 2)$ generators $M_{\mu\nu}$, translations P_μ , conformal boosts K_μ , dilatation D and the $SO(1, 1)$ generator A . The nonvanishing commutators for this bosonic subalgebra are

$$[M_{\mu\nu}, M_{\rho\sigma}] = \eta_{\nu\rho}M_{\mu\sigma} - \eta_{\nu\sigma}M_{\mu\rho} - \eta_{\mu\rho}M_{\nu\sigma} + \eta_{\mu\sigma}M_{\nu\rho},$$

$$[M_{\mu\nu}, P_\rho] = \eta_{\nu\rho}P_\mu - \eta_{\mu\rho}P_\nu,$$

$$[M_{\mu\nu}, K_\rho] = \eta_{\nu\rho}K_\mu - \eta_{\mu\rho}K_\nu,$$

$$[P_\mu, K_\nu] = 2(\eta_{\mu\nu}D - M_{\mu\nu}),$$

$$[P_\mu, D] = P_\mu, \quad [K_\mu, D] = -K_\mu.$$

The fermionic generators are the supersymmetry generators Q_\pm and the special supersymmetry generators S_\pm , all of which are real two component pseudo Majorana–Weyl spinors. The nonvanishing (anti)commutators involving the spinorial generators are

$$\{Q_-, Q_+\} = -2(P_- \gamma^\mu C)P_\mu,$$

$$\{S_+, S_-\} = 2(P_+ \gamma^\mu C)K_\mu,$$

$$\{S_\pm, Q_\pm\} = P_\pm(2D + \gamma^{\mu\nu}M_{\mu\nu} \pm A)C,$$

$$[P_\mu, S_\pm] = -\gamma_\mu Q_\pm, \quad [K_\mu, Q_\pm] = \gamma_\mu S_\pm,$$

$$[M_{\mu\nu}, Q_\pm] = -\frac{1}{2}\gamma_{\mu\nu}Q_\pm, \quad [M_{\mu\nu}, S_\pm] = -\frac{1}{2}\gamma_{\mu\nu}S_\pm,$$

$$[D, Q_\pm] = -\frac{1}{2}Q_\pm, \quad [D, S_\pm] = \frac{1}{2}S_\pm,$$

$$[A, Q_\pm] = \pm 3Q_\pm, \quad [A, S_\pm] = \mp 3S_\pm,$$

where $P_\pm = \frac{1}{2}(1 \pm \gamma_5)$.

References

- [1] H. Ooguri and C. Vafa, *Mod. Phys. Lett. A* 5 (1990) 1389; *Nucl. Phys. B* 361 (1991) 469; *B* 367 (1991) 83.
- [2] R.S. Ward, in: *Field theory, quantum gravity and strings*, Vol. 2, eds. H.J. De Vega and N. Sanchez, p. 106.
- [3] S.J. Gates Jr., H. Nishino and S.V. Ketov, *Extended supersymmetry and self-duality in 2+2 dimensions*, preprint UMDEPP 92-187; *Supersymmetric self-dual Yang–Mills and supergravity as background of Green–Schwarz superstring*, preprint UMDEPP 92-171; *Majorana–Weyl spinors and self-dual gauge fields in 2+2 dimensions*, preprint UMDEPP 92-163.
- [4] M.F. Sohnius and P.C. West, *Phys. Lett. B* 105 (1981) 353.
- [5] S. Ferrara, S. Sabharwal and M. Villasante, *Phys. Lett. B* 205 (1988) 302; S. Cecotti, S. Ferrara, M. Porrati and S. Sabharwal, *Nucl. Phys. B* 306 (1988) 160.
- [6] M. de Roo, A. Wiedemann and E. Zijlstra, *Class. Quantum Grav.* 7 (1990) 1181.
- [7] T. Kugo and P.K. Townsend, *Nucl. Phys. B* 221 (1983) 357.
- [8] E. Bergshoeff, E. Sezgin and P.K. Townsend, *Phys. Lett. B* 169 (1986) 191.
- [9] L.J. Romans, unpublished.